

On Computable Coordinate Systems: Definition, Operations, and Computer Code Implementation

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Abstract: Matrix operations are often used in coordinate system transformations, while tensor operations are utilized in more general situations. The former belongs to the category of "raw" mathematical objects, which are not designed for coordinate system transformations and have problems with redundancy and convenience in the calculation process. The latter is difficult to master and challenging to use numerically. This paper introduces a specialized structure object for coordinate system transformation as a substitute for matrix and tensor operations.

1. Introduction

Coordinate systems have always been the most abstract and difficult mathematical and physical objects, and the development of tensor operations has made them even more abstract, requiring years of training to barely master. Therefore, it is necessary to explore a simplified method to address the challenges of coordinate system operations. Although matrix operations are commonly used in coordinate system transformations, they are not designed for this purpose and are too mathematical and vague in meaning. Tensors, on the other hand, are too abstract and difficult to master, and it is difficult for computers to quantify them. A concept specifically designed for coordinate system transformation is needed.

2. Concept

The concept of a coordinate system and the concept of operable coordinates do not come from matrices, but from the use of a ruler. There are two uses for a ruler: 1) Defining a length on the ruler, such as 1 cm, and mapping it onto a physical entity for use, commonly used in the process of creating a physical object from a design.

2) Measuring the length value of a physical entity. This can be extended to general cases, including the measurement of all physical quantities.

Imagine a one-dimensional coordinate system, which corresponds to a ruler. If the ruler uses different units, such as centimeters, and measures a value \mathbf{v} , to convert it to a universal standard unit, a multiplication is needed:

$$\mathbf{v0} = \mathbf{v} * \text{cm}$$

Here, \mathbf{v} is the reading on the ruler, and $\mathbf{v0}$ is the length value in a universal unit.

In contrast, if we use a ruler to measure the length of a physical entity, it can be expressed as a division operation:

$$\mathbf{v} = \mathbf{v0} / \text{cm}$$

Here, \mathbf{v} is the reading on the ruler, and $\mathbf{v0}$ is the length value in a universal unit.

If we extend this concept to two or three dimensions, this multi-dimensional ruler becomes a coordinate system.

3. Design

A mathematical object called "coord" has been designed using group theory to facilitate arithmetic operations. The structure of the coordinate system is defined as follows: a three-dimensional coordinate system comprises an origin, three directional axes, and three scaling components, which represent translation, rotation, and scaling transformations. We will utilize mathematical expressions that can be overloaded with operators for ease of programming.

a) Definition of coordinate system structure

The coordinate system in three-dimensional space consists of an origin plus three direction axes and three scaling components, corresponding to the three transformations of **translation, rotation, and scaling** respectively (C++ version):

```
struct coord
{
    vec3 ux, uy, uz; // Three unit basis vectors
    vec3 s;           // Scaling
    vec3 o;           // Origin Position
}
```

Construct a coordinate system object (C++ version).

By axes:

```
coord(vec3 ux, vec3 uy, vec3 uz)
coord(vec3 ux, vec3 uy)
```

By Euler angles:

```
coord(float pitch, float yaw, float roll)
```

(Origin and scaling can be directly set)

b) Multiplication: Define a vector in a certain coordinate system

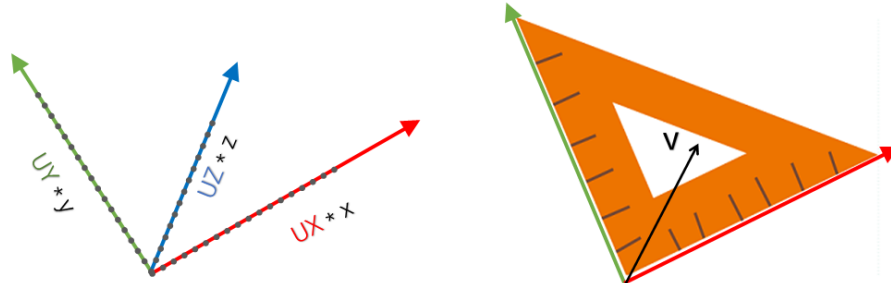


Figure 1: Define a vector in a coordinate system, with each component defined on its respective coordinate axis.

Use multiplication to transform a vector from the local coordinate system to the parent coordinate system and to merge coordinate systems.

For example: V_0 is defined in the world coordinate system C_0 , and V_1 is defined in the C_1 coordinate system.

$$V_0 = V_1 * C_1$$

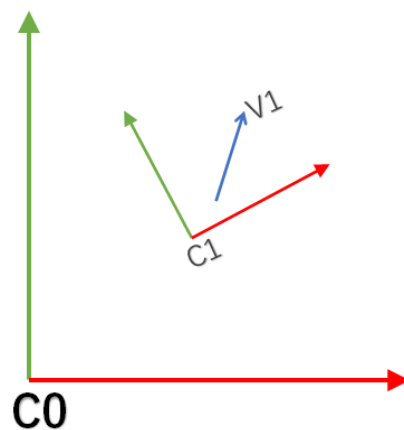


Figure 2: Vector V_1 is defined in coordinate system C_1 , with C_1 being the parent coordinate system of C_0 .

C_1 , C_2 , and C_3 are local coordinate systems in three scene nodes, and the parent-child direction is:

$$V_0 = V_3 * C_3 * C_2 * C_1 = V_3 * (C_3 * C_2 * C_1)$$

Multiplication after swapping with vectors:

$$\mathbf{V} * \mathbf{C} \neq \mathbf{C} * \mathbf{V}$$

Define the coordinate system and multiply the vector:

$$\mathbf{C} * \mathbf{V} = \text{Lerp}(\text{ZERO}, \mathbf{C}, \mathbf{V})$$

Here, Lerp is a linear interpolation function, ZERO is the zero coordinate system.

The multiplication of two vectors can be defined as follows:

$$\mathbf{V1} * \mathbf{V2} = \mathbf{ONE} * \mathbf{C1} * \mathbf{ONE} * \mathbf{C2} = \mathbf{ONE} * \mathbf{C1} * \mathbf{C2}$$

Here, **ONE** is the unity vector:

$$\mathbf{V} * \mathbf{ONE} = \mathbf{V}$$

$$\mathbf{C} * \mathbf{ONE} = \mathbf{C}$$

c) Division: Measure a vector using a certain coordinate system

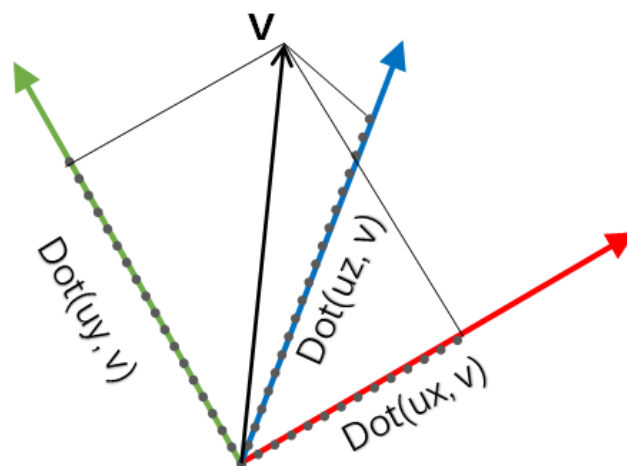


Figure 3: Measuring a vector V using a coordinate system, with each component value being the projection of V onto each coordinate axis.

Use division to project from the parent coordinate system to the local coordinate system. For example:

V0 is defined in the world coordinate system C0, and V1 is defined in the C1 coordinate system:

$$\mathbf{V1} = \mathbf{V0} / \mathbf{C1}, \mathbf{V0} = \mathbf{V1} * \mathbf{C1}$$

V2 is defined in the C2 coordinate system:

$$\mathbf{V2} = \mathbf{V0} / \mathbf{C2} = \mathbf{V1} * \mathbf{C1} / \mathbf{C2}$$

C1, C2, and C3 are local coordinate systems in three scene nodes, and the parent-child direction is:

$$\mathbf{V3} = \mathbf{V0} / \mathbf{C1} / \mathbf{C2} / \mathbf{C3} = \mathbf{V0} / (\mathbf{C3} * \mathbf{C2} * \mathbf{C1})$$

The division of two vectors can be defined as follows:

$$\mathbf{V1} / \mathbf{V2} = \mathbf{ONE} * \mathbf{C1} / (\mathbf{ONE} * \mathbf{C2}) = \mathbf{ONE} * \mathbf{C1} / \mathbf{C2}$$

Here, **ONE** is the unity vector,

$$\mathbf{V} / \mathbf{ONE} = \mathbf{V}$$

$$\mathbf{C} / \mathbf{ONE} = \mathbf{C}$$

In addition, due to the lack of multiplication commutativity, the division operation in the coordinate system has two types of division: left division is defined as:

$$\mathbf{C1} \setminus \mathbf{C2} = \mathbf{I} / \mathbf{C1} * \mathbf{C2}$$

Here, the left division symbol "\" is introduced for comparison with the commonly used right division "/". I is the identity coordinate system.

However, it is recommended to write the equation in the form on the right side to avoid confusion during calculations.

Of course, the exponential form below can also be used:

$$C1 \setminus C2 = C1^{(-1)} * C2$$

$$\text{Here, } C1^{(-1)} = \mathbf{ONE} / C1$$

d) Usual Application Scenarios

1. For example, a vector V_w in world space is converted to the local coordinate system C :

$$V_L = V_w / C$$

$$\text{Conversely: } V_w = V_L * C$$

2. Conversion between world coordinate system and local coordinate system

$$C = C3 * C2 * C1$$

$$V_w = V_L * C, \quad V_L = V_w / C$$

3. Multiple node hierarchy use

Vector V_5 is defined in the coordinate system of node 5, in the parent node at the level of node 2:

$$V_2 = V_5 * C_5 * C_4 * C_3$$

Each coordinate system is a local coordinate system

$$\text{Conversely: } V_5 = V_2 / C_3 / C_4 / C_5$$

4. Conversion between parallel coordinate systems

$C0\{ // C0$ under the parent coordinate system

C1, C2} // Two flat sub-coordinate systems

Convert a vector from C1 to C2 coordinate systems

$$\mathbf{V}_2 = \mathbf{V}_1 * C1 / C2$$

e) Addition and subtraction

When calculating the difference between two vectors:

$$\Delta \mathbf{V} = \mathbf{V}_1 - \mathbf{V}_2$$

$$\text{Let: } \mathbf{V}_1 = \mathbf{V} * C1, \mathbf{V}_2 = \mathbf{V} * C2$$

$$\Delta \mathbf{V} = \mathbf{V} * (C1 - C2)$$

Here defines an addition and subtraction operation for basic addition and subtraction of vectors:

$$\mathbf{V}_1 + \mathbf{V}_2 = \mathbf{V} * (C1 + C2)$$

$$\mathbf{V}_1 - \mathbf{V}_2 = \mathbf{V} * (C1 - C2)$$

To facilitate our operations, we establish a principle: for objects of different types, the operation structure remains the same as the object on the left side of the operator.

For example:

$$\mathbf{V}(\text{vector}) = \mathbf{V}_c(\text{vector}) * C(\text{coordinate})$$

$$C1(\text{coordinate}) = C2(\text{coordinate}) * \mathbf{V}(\text{vector})$$

4. Advanced Application Scenarios

a) Coordinate System Differentiation

Constructing a differential coordinate system where a vector can be defined, and the components of the vector correspond to derivatives. This allows for a more intuitive understanding of differentiation operations in space. Additionally, we can perform vector operations in the differential coordinate system, multiplying the resulting vector with the differential coordinate system

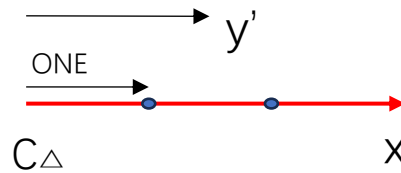


Figure 4: Defining the derivative vector in a differential coordinate system.

to obtain a differential vector. This simplifies the expression method.

$$\Delta \mathbf{x} = \mathbf{ONE} * C\Delta$$

$$\Delta \mathbf{y} = \mathbf{y}' * C\Delta$$

$$\Delta \mathbf{y} / \Delta \mathbf{x} = \mathbf{y}'$$

Differentiation of the coordinate system in space corresponds to three common forms:

1) Gradient: $\nabla f = \mathbf{U} * df * C_u / (l_c * d\mathbf{xyz})$

Here, \mathbf{U} represents the principal direction of change, defined in the local coordinate system C_u , df represents the amount of change, C_u represents the local coordinate system, and $d\mathbf{xyz}$ represents the vector element defined in

the world coordinate system, lc is the default world coordinate system. The multiplication of lc and $dxyz$ represents a differential coordinate system.

2) Divergence: $\nabla \cdot \mathbf{F} = d\mathbf{F} / (lc * d\mathbf{xyz}) \cdot lc$

Here, $d\mathbf{xyz}$ represents the vector element defined in the world coordinate system, lc is the default world coordinate system. The multiplication of lc and $d\mathbf{xyz}$ represents a differential coordinate system.

The dot product operation can be appropriately defined between vectors and coordinate systems.

3) Curl: $\nabla \times \mathbf{F} = d\mathbf{F} / (lc * d\mathbf{xyz}) \times lc$

Here, $d\mathbf{xyz}$ represents the vector element defined in the world coordinate system, lc is the default world coordinate system. The multiplication of lc and $d\mathbf{xyz}$ represents a differential coordinate system.

The cross product operation can be appropriately defined between vectors and coordinate systems.

In order to facilitate numerical calculations, we set a sufficiently small scaling factor ϵ for the global coordinate system, where we can approximately perform linear operations and ignore higher-order terms.

On this scale, we can use the unit one to replace the differential quotient factor of derivatives, thereby directly using differentials instead of derivatives, achieving the goal of simplifying expressions.

Therefore, the gradient expression mentioned above can be written as:

$$G_{12} = C_1 \setminus C_2 - I$$

$$\Delta \mathbf{V} = \mathbf{V1} * G12$$

Here C1 and C2 are two adjacent coordinate systems, I is the identity coordinate system. $\mathbf{V12}$ is the differential change between the two coordinate systems C1 and C2.

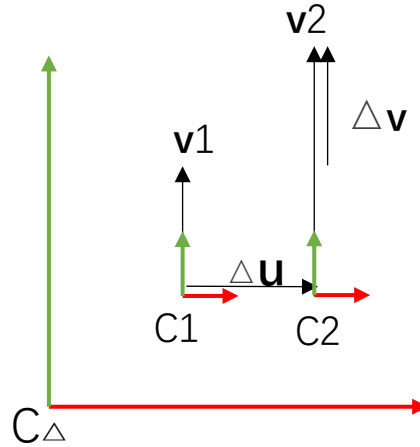


Figure 5: Defining the derivative vector in a differential coordinate system.

In the differential coordinate system, let C_{Δ} be the world coordinate system, and C1, C2 be the child coordinate systems. Vectors $\mathbf{v1}$ and $\mathbf{v2}$ are defined as:

$$\mathbf{v1} = \mathbf{ONE} * C1, \mathbf{v2} = \mathbf{ONE} * C2$$

$$\Delta \mathbf{v} = \mathbf{v2} - \mathbf{v1} = \mathbf{ONE} * (C2 - C1)$$

Assuming $\Delta \mathbf{u}$ represents the positional vector difference from C1 to C2:

$$\Delta \mathbf{u} = \mathbf{ONE} * C_{\Delta}$$

Considering that \mathbf{ONE} does not work in vector and coordinate system operations, it can be removed:

$$\text{The derivative: } \Delta \mathbf{v} / \Delta \mathbf{u} = (C2 - C1) / C_{\Delta}$$

Suppose we need a form:

$$\mathbf{v1} * G12 = \mathbf{v2} - \mathbf{v1} = \mathbf{ONE} * (C2 - C1).$$

$$G_{12} = C_1 \setminus C_2 - I$$

Therefore: $\Delta \mathbf{v} / \Delta \mathbf{u} = \mathbf{v}_1 * G_{12} / C_{\Delta}.$

If we choose a suitable differential coordinate system $C_{\Delta} = I$:

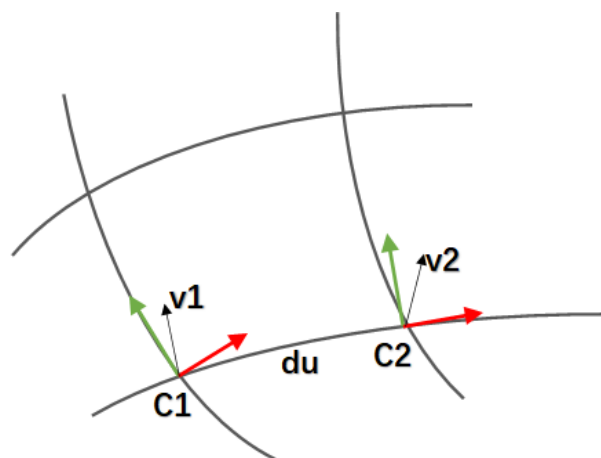
$$\Delta \mathbf{v} / \Delta \mathbf{u} = \mathbf{v}_1 * G_{12} = \mathbf{v} * G_u$$

By selecting a suitable differential coordinate system, we can simplify the expression for vector differentiation.

b) Applications in the Field of Differential Geometry

1) Movement of Vectors

Assuming a flat space, a natural coordinate system is defined in which a vector V can move freely without changing. When observed under a curved space coordinate system, V varies at different points, indicating that the coordinate system is position-dependent. Let (1) and (2) be two adjacent points with vectors V_1 and V_2 at each point, corresponding to coordinate systems C_1 and C_2 . Then, we have:



$$\mathbf{V} = \mathbf{V}_1 * C_1 = \mathbf{V}_2 * C_2 \Rightarrow$$

$$\mathbf{V}_2 = \mathbf{V}_1 * C_1 / C_2,$$

$$\text{Let: } R_{12} = C_1 / C_2 \Rightarrow$$

$$\mathbf{V}_2 = \mathbf{V}_1 * R_{12}$$

Figure 6: Defining two coordinate systems on a surface space, where vector V_1 is translated from one point to another, resulting in V_2 .

2) Exponential Operation of Coordinate Systems

When a coordinate system is translated along the arc of a differential space, it often involves rotation. To calculate the derivative of rotation with respect to translation, the exponential notation can be utilized. A coordinate system can be represented in exponential form as:

$$C = e^{(\mathbf{n})}$$

$$\mathbf{n} = \mathbf{axis} * \text{angle}$$

where \mathbf{n} represents the rotation axis multiplied by the angle of rotation.

Utilizing the Taylor series expansion, we can approximate the exponential form as:

$$e^{(\mathbf{n})} \approx 1 + (\mathbf{n}) - \dots = \cos(\text{angle}) + \sin(\text{angle}) * \mathbf{axis}$$

This approximation allows us to calculate the derivative of rotation with respect to translation by replacing division operations in rotation computations with the exponential form.

Considering the coordinate systems at the two ends of a differential space arc, u_1 and u_2 , we have:

$$C_1 = e^{(\mathbf{n}_1)}, C_2 = e^{(\mathbf{n}_2)}$$

$$R_{12} = C_1 / C_2 = e^{(\mathbf{n}_1 - \mathbf{n}_2)}$$

let $d\mathbf{n} = \mathbf{n}_1 - \mathbf{n}_2$, $d\mathbf{u} = \mathbf{u}_1 - \mathbf{u}_2$:

$$\text{Derivative } R_u = e^{(d\mathbf{n} / d\mathbf{u})}$$

An arbitrary differential vector can be defined as:

$$d\mathbf{w} = d\mathbf{u} * R_u$$

So to move in any direction of a differential vector:

$$\mathbf{V}_2 = \mathbf{V}_1 * \mathbf{R}_u \wedge (d\mathbf{w} / \mathbf{R}_u)$$

This quotient represents the variation of two coordinate system objects C1, C2 in the direction of the unit differential vector \mathbf{u} .

In addition, the change in two vectors can be computed using the gradient of the coordinate system:

$$\Delta \mathbf{V} = \mathbf{V}_2 - \mathbf{V}_1 = \mathbf{V}_1 * \mathbf{G}_u * (d\mathbf{w} / \mathbf{R}_u)$$

where $\mathbf{G}_u = \mathbf{C}_1 / \mathbf{C}_2 - \mathbf{I}$, \mathbf{I} is the unit coordinate system.

5. Curvature Calculation

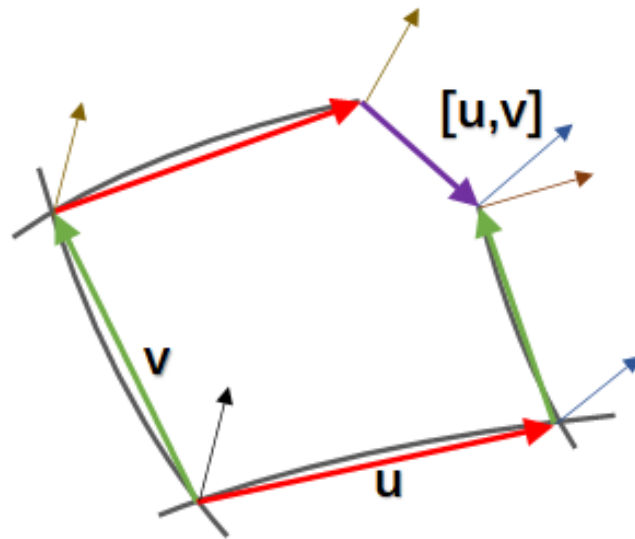


Figure 7: In a small element of the surface space, when a vector is translated along two perpendicular directions, the final result may differ depending on the path taken.

The coordinate system object enables the conversion of vectors from natural coordinates to curved coordinates. Curvature can be calculated by comparing the difference in vector translation along the $u \rightarrow v$ and $v \rightarrow u$ paths using the intrinsic gradient operators \mathbf{G}_u and \mathbf{G}_v , which represent the rotational change

gradients of the frame along the \mathbf{u} and \mathbf{v} directions.

Using coordinate system operations, spatial curvature can be directly computed. The coordinate-equivalent representation of the Riemann curvature tensor in the (\mathbf{u}, \mathbf{v}) coordinate system is given by:

$$\mathbf{R}_{uv} = [\mathbf{G}_u, \mathbf{G}_v] - \mathbf{G}[\mathbf{u}, \mathbf{v}] = \mathbf{G}_u \mathbf{G}_v - \mathbf{G}_v \mathbf{G}_u - \mathbf{G}[\mathbf{u}, \mathbf{v}]$$

where:

- $\mathbf{G}_u = (\mathbf{C}_u / \mathbf{C}_0) - \mathbf{I}$ is the connection along the \mathbf{u} -direction
- $\mathbf{G}_v = (\mathbf{C}_v / \mathbf{C}_0) - \mathbf{I}$ is the connection along the \mathbf{v} -direction
- \mathbf{I} is the identity coordinate system
- \mathbf{C}_0 is the initial reference coordinate system

Connection Vector: $[\mathbf{u}, \mathbf{v}]$ (Lie Bracket Operation)

Let $\mathbf{W} = [\mathbf{u}, \mathbf{v}]$, where \mathbf{W}_u and \mathbf{W}_v are the components of \mathbf{W} . Using the coordinate system and vector multiplication defined earlier, the change of the gradient coordinate system along \mathbf{W} is calculated as:

$$\mathbf{G}[\mathbf{u}, \mathbf{v}] = \mathbf{G}_u \cdot \mathbf{W}_u + \mathbf{G}_v \cdot \mathbf{W}_v$$

6. Combination with Lie groups, Lie algebras

The rotation matrix R , which describes the rotation of an object in three-dimensional space, can be classified as a special type of matrix belonging to a mathematical structure known as a Lie group. A Lie group is a mathematical object that possesses both continuous and group structure, and it finds

extensive applications in physics and geometry.

In a similar vein, our coordinate system C can be considered as an element of the Lie group. This implies that we can use the multiplication operation of the Lie group to represent transformations between coordinate systems. Specifically, the multiplication operation of our coordinate system can be expressed as matrix multiplication, with the identity element I representing the identity transformation.

Hence, we can represent our coordinate system as:

$$C \in \{I, (*)\}$$

where $(*)$ denotes the multiplication operation of the coordinate system.

Utilizing the concept of Lie groups, we can derive the expression for the vector **cross product in terms of coordinate systems. Let \mathbf{v}_1 and \mathbf{v}_2 be two vectors**, and C_1 and C_2 be the corresponding elements of the coordinate system. The cross product of \mathbf{v}_1 and \mathbf{v}_2 can be written as:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{v} * C_1 \times (\mathbf{v} * C_2)$$

Here, \mathbf{v} represents the coordinate system representation of vectors \mathbf{v}_1 and \mathbf{v}_2 , which can be expressed as the unity vector **ONE**. Simplifying the equation, we obtain:

$$\mathbf{v}_1 \times \mathbf{v}_2 = \mathbf{ONE} * (C_1 \times C_2)$$

To further describe the cross product operation of coordinate systems, we introduce the concept of Lie algebra. We define the coordinate system cross product operation using Lie algebra brackets:

$$[C1, C2] = C1 * C2 - C2 * C1$$

This expression represents the operation of multiplying C1 and C2 and subtracting the result from the multiplication of C2 and C1, as defined by Lie algebra. This abstract representation provides a concise way to describe the operations between coordinate systems.

7. Conclusion

In conclusion, the use of coordinate system objects offers a simplified approach to linear operations in local space. By appropriately designing operations for vector movement and rotation, we can effectively streamline complex calculations. Moreover, by adhering to the principles of group theory, a unified form can be achieved, enhancing the efficiency and coherence of the overall process.

Coordinate system objects find extensive application in scenarios that involve numerous linear operations, such as constraint calculations. By utilizing these objects, we can effectively manage and manipulate vectors, matrices, and tensors, seamlessly integrating them into a cohesive framework. This integration extends to the unification of Lie groups and Lie algebraic objects, which are combined into a cohesive ring structure.

The utilization of coordinate system objects, in conjunction with Lie groups and Lie algebra, allows for a comprehensive and unified approach to mathematical operations. This unified framework not only simplifies complex

calculations but also provides a consistent and coherent methodology for handling various mathematical entities. Overall, the integration of coordinate system objects with Lie groups and Lie algebraic objects offers a powerful and versatile tool for solving problems that involve extensive linear operations.

8. Partial code implementation

Implement using C++:

```
vec3 operator * (const vec3& p, const coord3& c)
{
    return c.ux * (c.s.x * p.x) + c.uy * (c.s.y * p.y) + c.uz * (c.s.z * p.z) + c.o;
}

coord3 operator * (const coord3& c1, const coord3& c2)
{
    coord3 rc;
    rc.ux = c1.ux.x * c2.ux + c1.ux.y * c2.uy + c1.ux.z * c2.uz;
    rc.uy = c1.uy.x * c2.ux + c1.uy.y * c2.uy + c1.uy.z * c2.uz;
    rc.uz = c1.uz.x * c2.ux + c1.uz.y * c2.uy + c1.uz.z * c2.uz;
    rc.s = c1.s * c2.s;
    rc.o = c1.o.x * c2.s.x * c2.ux + c1.o.y * c2.s.y * c2.uy + c1.o.z * c2.s.z * c2.uz + c2.o;
    return rc;
}

vec3 operator / (const vec3& p, const coord3& c)
{
    vec3 v = p - c.o;
    return vec3( v.dot(c.ux) / c.s.x, v.dot(c.uy) / c.s.y, v.dot(c.uz) / c.s.z );
}

coord3 operator / (const coord3& c1, const coord3& c2)
{
    coord3 rc;
    rc.ux = vec3(c1.ux.dot(c2.ux), c1.ux.dot(c2.uy), c1.ux.dot(c2.uz));
    rc.uy = vec3(c1.uy.dot(c2.ux), c1.uy.dot(c2.uy), c1.uy.dot(c2.uz));
    rc.uz = vec3(c1.uz.dot(c2.ux), c1.uz.dot(c2.uy), c1.uz.dot(c2.uz));
    rc.s = c1.s / c2.s;
    rc.o = c1.o - c2.o;
    rc.o = vec3(rc.o.dot(c2.ux) / c2.s.x, rc.o.dot(c2.uy) / c2.s.y, rc.o.dot(c2.uz) / c2.s.z);
    return rc;
}

coord3 operator + (const coord3& c1, const coord3& c2)
{

```

```

    coord3 rc;
    rc.ux = c1.VX() + c2.VX();
    rc.uy = c1.VY() + c2.VY();
    rc.uz = c1.VZ() + c2.VZ();
    rc.norm();
    rc.o = o + c2.o;
    return rc;
}

coord3 operator - (const coord3& c1, const coord3& c2)
{
    coord3 rc;
    rc.ux = c1.VX() - c2.VX();
    rc.uy = c1.VY() - c2.VY();
    rc.uz = c1.VZ() - c2.VZ();
    rc.norm();
    rc.o = o - c2.o;
    return rc;
}

```

Reference:

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